Chapter 10

INEQUALITIES

We deal only with real numbers in this chapter.

10.1 ELEMENTARY PROPERTIES

If a and b are real numbers, $a < b$ (read "a is less that b") is defined to mean that $b - a$ is positive. This definition and the following three properties can be used in proving elementary properties of inequalities:

(1) **Closure of the positive numbers**. If a and b are positive numbers, then $a + b$ and ab are positive numbers.

(2) **Trichotomy**. One and only one of the following is true for a given real number a: (a) a is zero; (b) a is positive; (c) $-a$ is positive.

(3) **Roots**. If p is a positive number and n is a positive integer, then there is exactly one positive number r such that $r^n = p$. (This number r is called the positive nth root of p or the principal nth root of p.)

We can write the statement $a < b$ in the form $b > a$ (read "b is greater than a"). The notation $a \leq b$ (read "a is less than or equal to b") means that either $a < b$ or $a = b$, and $b \geq a$ is defined analogously. The notation $x < y < z$ or $z > y > x$ means that $x < y$ and $y < z$ are true simultaneously.

As mentioned is Section 8.4, the **absolute value** of a real number x is written as $|x|$ and is defined as follows: If $x \geq 0$, then $|x| = x$; if $x < 0$, then $|x| = -x$.

Example 1. Show that if $x < y$ and $y < z$, that is $x < y < z$, then $x < z$.

Solution: If $x < y$ then $y - x = p$, a positive number, and similarly $y < z$ implies that $z - y = q$ where q is positive. Hence $(z - y) + (y - x) = q + p$, or

$$z - x = q + p.$$

Since $z - x$ is the sum of the positive numbers q and p, it is positive by the closure property and thus $x < z$ by the definition.

Example 2. Show that if $x < y$ and $p > 0$, then $px < py$.

Solution: If $x < y$, then $y - x$ is positive and, by closure, the product $p(y - x)$ of positive numbers is positive; that is, $py - px$ is positive. Now we have $px < py$ by definition.

Example 3. Show that if m and n are integers and $m < n$, then $n - m \geq 1$.

Solution: Since $m < n$, it follows that $n - m$ is positive. Since the difference of integers is an integer and the least positive integer is 1, $n - m$ is at least 1; that is $n - m \geq 1$.

Problems for Section 10.1

R 1. (a) Show that if $x < y$, then $x + z < y + z$.
(b) Show that if $x < y$, then $x - w < y - w$.

R 2. Show that if $x < y$ and $q < 0$, then $qx > qy$.

R 3. (a) Show that if $x > 0$ or $x < 0$, then $x^2 > 0$.
(b) Show that for all real x, $x^2 \geq 0$.
(c) Show that $1 > 0$.

R 4. (a) Show that if $x > 0$, then $\frac{1}{x} > 0$, and if $x < 0$, then $\frac{1}{x} < 0$.

(b) Show that if $0 < x < y$ or $x < y < 0$, then $\frac{1}{x} > \frac{1}{y}$.

R 5. Show the following:

(a) If $0 < x < y$ and n is a positive integer, then $x^{2n-1} < y^{2n-1}$.
(b) If $x < 0 < y$ and n is a positive integer, then $x^{2n-1} < y^{2n-1}$.
(c) If $x < y < 0$ and n is a positive integer, then $x^{2n-1} < y^{2n-1}$.

R 6. (a) Show that if $0 < x < y$ and n is a positive integer, then $x^{2n} < y^{2n}$.
(b) Show that if $y < x < 0$ and n is a positive integer, then $x^{2n} < y^{2n}$.

R 7. Show that if n is a positive integer and $x^{2n-1} < y^{2n-1}$, then $x < y$. (See Problem 5.)

R 8. (a) Show that if $x^{2n} < y^{2n}$ and $y > 0$, then $-y < \pm x < y$. (See Problem 6.)
(b) Show that if $x^{2n} < y^{2n}$ and $y < 0$, then $y < \pm x < -y$.
(c) Use Parts (a) and (b) to show that if $x^{2n} < y^{2n}$, then $-|y| < \pm x < |y|$.

R 9. Prove the following by mathematical induction:

(a) If $a_1, a_2, ..., a_n$ are positive, so is $a_1 + a_2 + ... + a_n$.
(b) If $a_1, a_2, ..., a_n$ are positive, so is $a_1 a_2 ... a_n$.
(c) If $a_1 < b_1, a_2 < b_2, ..., a_n < b_n$, then $a_1 + a_2 + ... + a_n < b_1 + b_2 + ... + b_n$.
(d) If $0 < a_1 < b_1, 0 < a_2 < b_2, ..., 0 < a_n < b_n$, then $a_1 a_2 ... a_n < b_1 b_2 ... b_n$.

R 10. Show that:

(a) $x^2 - 2xy + y^2 \geq 0$.
(b) $x^2 + y^2 \geq 2xy$.

(See Problem 3.)

11. Given that $x \neq y$, show that:

(a) $x - y \neq 0$.
(b) $x^2 - 2xy + y^2 > 0$.
(c) $x^2 + y^2 > 2xy$.

12. Find all the integers n such that $2n^2 - 3 < 8n$.

13. (a) Given that $0 < a < b$, show that $a^2 < ab < b^2$.
(b) Given that $0 < a < b$, show that $3a^2 < a^2 + ab + b^2 < 3b^2$.

14. (a) Given that $a < b$, show that $a < \dfrac{a+b}{2} < b$.

(b) Given that $a < b$, show that $a < \dfrac{2a+b}{3} < \dfrac{a+2b}{3} < b$.

15. Given that $0 < x < y$, show the following:

(a) $\dfrac{x-1}{x} < \dfrac{y-1}{y}$.

(b) $\dfrac{x}{x+1} < \dfrac{y}{y+1}$.

16. Given that $0 < x \leq y \leq z$ and that $x + y > z$, show that $\dfrac{x}{x+1} + \dfrac{y}{y+1} > \dfrac{z}{z+1}$.

17. (a) Given that $0 < a < b \leq c$, show that $c > \dfrac{a+b}{2}$.

(b) Given that $0 < a \leq b < c$, show that $c > \dfrac{a+b}{2}$.

(c) Given that $0 < a \leq b \leq c$ and a < c, show that $c > \dfrac{a+b}{2}$.

R 18. Given that $0 < a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_{n-1} \leq a_n$ and $a_1 < a_n$, show that $a_n > (a_1 + a_2 + \ldots + a_{n-1})/(n - 1)$.

19. Find integers a, b, and c such that $0 < a < b < c$, $a + b > c$, and c is as small as possible.

20. Let m and n be positive integers, and let 1, m, and n be the lengths of the sides of a triangle. Show that $m = n$.

21. Given that $x > 0$ and $y > 0$, show that $(x + y)^n > x^{n-1}(x + ny)$ for all integers $n \geq 2$.

22. Given that $1 + x \geq 0$, prove by mathematical induction that $(1 + x)^n \geq 1 + nx$ for all positive integers n.

23. Prove that $2\sqrt{x} < \dfrac{1}{\sqrt{x+1} - \sqrt{x}}$ for all positive real numbers x.

24. Prove that $\dfrac{2}{3}n\sqrt{n} < \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}$ for all positive integers n.

25. Prove that $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} < \dfrac{(4n+3)\sqrt{n}}{6}$ for all positive integers n.

26. Use the fact that $1<b$ and $x<y$ imply $b^x<b^y$ to prove the inequalities $\sqrt{2} \leq a_n < a_{n+1} < 2$ for the sequence a_1, a_2, ... defined by

$$a_1 = \sqrt{2},\ a_2 = \left(\sqrt{2}\right)^{a_1},\ \ldots,\ a_{n+1} = \left(\sqrt{2}\right)^{a_n},\ \ldots .$$

27. Use the fact that $0 < b < 1$ and $x < y$ imply $b^x > b^y$ to prove the inequalities

$$u_1 < u_2,\ u_2 > u_3,\ \ldots,\ u_{2k-1} < u_{2k},\ u_{2k} > u_{2k+1}, \ldots$$

for the sequence u_1, u_2, ... defined by

$$u_1 = \frac{1}{\sqrt{2}},\ u_2 = \left(\frac{1}{\sqrt{2}}\right)^{u_1},\ \ldots,\ u_{n+1} = \left(\frac{1}{\sqrt{2}}\right)^{u_n},\ \ldots .$$

10.2 FURTHER INEQUALITIES

In this section we develop a technique for investigating the range of values assumed by a quadratic function. In subsequent work we shall assume as known the results of the examples in Section 10.1 and of Problems 1 to 8 in Section 10.1.

Example: Let $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. Let D be the ***discriminant*** $b^2 - 4ac$. Show that if $D > 0$, then $f(x)$ takes on both positive and negative values.

Solution: Completing squares, we obtain

$$\begin{aligned} f(x) &= ax^2 + bx + c = \frac{4a^2x^2 + 4abx + 4ac}{4a} \\ &= \frac{4a^2x^2 + 4abx + b^2 - (b^2 - 4ac)}{4a} \\ &= \frac{(2ax+b)^2 - D}{4a}. \end{aligned}$$

If $x = -b/2a$, $2ax + b = 0$, and so $f(-b/2a) = -D/4a$. We first consider the case in which $a > 0$. This and $D > 0$ imply that $f(x) = -D/4a < 0$. We wish to show that $f(x)$ also takes on positive values.

We consider values of x greater that $(-b+\sqrt{D})/2a$. Then

$$\begin{aligned} x &> \frac{-b+\sqrt{D}}{2a} \\ 2ax &> -b+\sqrt{D} \\ 2ax + b &> \sqrt{D} > 0 \\ (2ax+b)^2 &> D \\ (2ax+b)^2 - D &> 0 \\ f(x) = \frac{(2ax+b)^2 - D}{4a} &> 0. \end{aligned}$$

Thus we have proved the desired result for the case $a > 0$. If $a < 0$, let $g(x) = -ax^2 - bx - ac$. Since the coefficient of x^2 in $g(x)$ is positive, $g(x)$ takes on both positive and negative values by the previous case. Then so does $f(x) = -g(x)$.

Problems for Section 10.2

1. Let a and b be real numbers. Prove that $a^2 + b^2 \geq 0$ and that $a^2 + b^2 = 0$ if and only if $a = b = 0$.

2. Let $c_1, c_2, \ldots, c_n$ be real numbers. Prove that $c_1^2 + c_2^2 + \cdots + c_n^2 \geq 0$ and $c_1^2 + c_2^2 + \cdots + c_n^2 = 0$ if and only if each $c_i = 0$.

3. Let $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a > 0$. Let D be the discriminant $b^2 - 4ac$. Show the following:

(a) If $D < 0$, then $f(x) > 0$ for all x, and $f(x) = 0$ has no real roots.
(b) If $D = 0$, then $f(x) \geq 0$ for all x, and $f(x) = 0$ has one real root.
(c) If $D > 0$, then $f(x) < 0$ for $\frac{-b - \sqrt{D}}{2a} < x < \frac{-b + \sqrt{D}}{2a}$, $f(x) > 0$ for $x < \frac{-b - \sqrt{D}}{2a}$ or $x > \frac{-b + \sqrt{D}}{2a}$, and $f(x) = 0$ has two real roots.

4. Let $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a < 0$. Let $D = b^2 - 4ac$. Show the following:

(a) If $D < 0$, then $f(x) < 0$ for all x, and $f(x) = 0$ has no real roots.

(b) If $D = 0$, then $f(x) \leq 0$ for all x, and $f(x) = 0$ has one real root.

(c) If $D > 0$, then $f(x) > 0$ for $\frac{-b - \sqrt{D}}{2a} > x > \frac{-b + \sqrt{D}}{2a}$, $f(x) < 0$ for $x > \frac{-b - \sqrt{D}}{2a}$ or $x < \frac{-b + \sqrt{D}}{2a}$, and $f(x) = 0$ has two real roots.

*5. Let $F_1, F_2, F_3, \ldots$ be the sequence of Fibonacci numbers 1, 1, 2, 3, 5, ... and let $R_n = F_{n+1}/F_n$ for $n = 1, 2, 3, \ldots$. Do the following:

(a) Show that $R_{n+1} = 1 + \frac{1}{R_n}$.
(b) Prove that $R_{2n-1} < R_{2n+1} < R_{2n}$ and $R_{2n+1} < R_{2n+2} < R_{2n}$ for all positive integers n, that is, that $R_1 < R_3 < R_5 < R_7 < \ldots < R_8 < R_6 < R_4 < R_2$.

10.3 INEQUALITIES AND MEANS

We recall that the arithmetic mean of $a_1, a_2, \ldots, a_n$ is

$$\frac{a_1 + a_2 + \cdots + a_n}{n}$$

and the geometric mean is

$$\sqrt[n]{a_1 a_2 \cdots a_n}.$$

We restrict $a_1, \ldots, a_n$ to be positive in discussing geometric means, since otherwise the definition might express the mean as an even root of a negative number.

We shall use A_n for the arithmetic mean of $a_1, a_2, \ldots, a_n$ and G_n for the geometric mean.

THEOREM: Let $a_1, a_2, ..., a_n$ be positive real numbers. Then

$$A_n \geq G_n$$

that is,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Proof: We proceed by mathematical induction. When $n = 1$, it is clear that $A_1 = a_1$ and $G_1 = a_1$. Hence $A_1 = G_1$, and the theorem holds for $n = 1$.

We next prove it for $n = 2$. Since a_1 and a_2 are positive, $\sqrt{a_1}$ and $\sqrt{a_2}$ exist in the real number system by the roots property of Section 10.1, and so

$$\left(\sqrt{a_1} - \sqrt{a_2}\right)^2 \geq 0$$

by Problem 3(b) of section 10.1. It follows that

$$\begin{aligned} a_1 - 2\sqrt{a_1}\sqrt{a_2} + a_2 &\geq 0 \\ a_1 + a_2 &\geq 2\sqrt{a_1}\sqrt{a_2} \\ \frac{a_1 + a_2}{2} &\geq \sqrt{a_1 a_2}. \end{aligned}$$

This is precisely the statement $A_2 \geq G_2$ of the theorem for $n = 2$.

We now assume the inequality true for $n = k$, that is, we assume that $A_k \geq G_k$, and with this as a basis shall prove that $A_{k+1} \geq G_{k+1}$.

If $a_1 = a_2 = ... = a_{k+1}$ then $A_{k+1} = a_1, G_{k+1} = a_1$, and so $A_{k+1} \geq G_{k+1}$. It remains to investigate the case in which the a_i are not all equal. Without loss of generality, we may assume that the a_i are numbered so that

$$0 < a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{k+1}.$$

The fact that the a's are not all equal implies that $a_1 < a_{k+1}$. It now follows from Problem 18 of Section 10.1 that $a_{k+1} > A_k$. Since

$$A_k = \frac{a_1 + a_2 + \cdots + a_k}{k}$$

we have $kA_k = a_1 + a_2 + ... + a_k$, and hence

$$\begin{aligned}A_{k+1} &= \frac{a_1 + a_2 + \cdots + a_k + a_{k+1}}{k+1}\\ &= \frac{kA_k + a_{k+1}}{k+1}\\ &= \frac{(k+1)A_k + (a_{k+1} - A_k)}{k+1}\\ &= A_k + \frac{a_{k+1} - A_k}{k+1}.\end{aligned}$$

Let $(a_{k+1} - A_k)/(k + 1) = p$. We have seen above that $a_{k+1} > A_k$; this implies that $p > 0$. Now $A_{k+1} = A_k + p$. We raise both sides of this equality to the $(k + 1)$th power, obtaining

$$\begin{aligned}(A_{k+1})^{k+1} &= (A_k + p)^{k+1}\\ &= (A_k)^{k+1} + (k+1)(A_k)^k p + \binom{k+1}{2}(A_k)^{k-1}p^2 + \cdots + p^{k+1}.\end{aligned}$$

Since $p > 0$ and $A_k > 0$, all the terms in the binomial expansion on the right side are positive. There are $k + 2$ terms in this expansion; hence there are at least 4 terms. Now $(A_{k+1})^{k+1}$ is greater than the sum of the first two terms:

$$(A_{k+1})^{k+1} > (A_k)^{k+1} + (k+1)p(A_k)^k.$$

Since $(k + 1)p = a_{k+1} - A_k$, this becomes

$$\begin{aligned}(A_{k+1})^{k+1} &> (A_k)^{k+1} + a_{k+1}(A_k)^k - (A_k)^{k+1}\\ (A_{k+1})^{k+1} &> a_{k+1}(A_k)^k.\end{aligned}$$

Having assumed above that $A_k \geq G_k$, we now have

$$\begin{aligned}&(A_{k+1})^{k+1} > a_{k+1}(A_k)^k \geq a_{k+1}(G_k)^k = a_{k+1}(a_1 a_2 \cdots a_k)\\ &(A_{k+1})^{k+1} > (G_{k+1})^{k+1}.\end{aligned}$$

From Problems 7 and 8, Section 10.1, it follows that $A_{k+1} > G_{k+1}$, and the theorem is proved.

We have actually proved more than is stated in the theorem; we have shown that $A_n > G_n$ unless all the a_i are equal.

There is a third type of mean that is used quite often: the **harmonic** **mean**. The harmonic mean of numbers a_1, a_2, ..., a_n, none of which is zero, is given by

$$H_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

In a **harmonic progression**, defined as a progression of the form

$$\frac{1}{a},\ \frac{1}{a+d},\ \frac{1}{a+2d},\ \ldots,\ \frac{1}{a+(n-1)d}$$

each term, except the end ones, is the harmonic mean of its two adjacent terms; that is, if a_1, a_2, ..., a_n are in harmonic progression, then

$$\frac{2}{\dfrac{1}{a_{k-1}} + \dfrac{1}{a_{k+1}}} = a_k.$$

The proof is left to the reader.

10.4 THE CAUCHY-SCHWARZ INEQUALITY

Another famous inequality is given various names in different texts, although in the United States it is usually referred to as the **Cauchy - Schwarz Inequality** (named for Augustin Cauchy, 1789-1857; and Hermann Amandus Schwarz, 1843-1921). Some call it the Schwarz Inequality, while others, including the Russians, call it the Cauchy-Buniakowski Inequality.

THEOREM: Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any real numbers. Then

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2,$$

that is,

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

Proof: We define a polynomial function $f(x)$ by

$$f(x) = (a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \ldots + (a_n x + b_n)^2.$$

Clearly $f(x)$ is positive or zero for all real numbers x, since it is a sum of squares. Now

$$\begin{aligned} f(x) &= (a_1^2 x^2 + 2a_1 b_1 x + b_1^2) + (a_2^2 x^2 + 2a_2 b_2 x + b_2^2) + \cdots + (a_n^2 x^2 + 2a_n b_n x + b_n^2) \\ &= (a_1^2 + a_2^2 + \cdots + a_n^2)x^2 + 2(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)x + (b_1^2 + b_2^2 + \cdots + b_n^2). \end{aligned}$$

Let $a_1^2 + a_2^2 + \cdots + a_n^2 = A,\ a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = B,$ and $b_1^2 + b_2^2 + \cdots + b_n^2 = C$ so that

$f(x) = Ax^2 + 2Bx + C$. Since $f(x) \geq 0$ for all x, the discriminant $D \leq 0$, since $D > 0$ implies that $f(x)$ is sometimes positive and sometimes negative. (See Section 10.2.) Now

$$D = (2B)^2 - 4AC = 4B^2 - 4AC \leq 0.$$

Hence $B^2 - AC \leq 0$, and so $AC \geq B^2$. Translating this back into our original notation, we have the Cauchy-Schwarz Inequality:

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2.$$

Examination of the above proof shows that

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) = \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

if and only if there is a fixed number x such that $a_i x + b_i = 0$ for all i, that is, the a_i and b_i are proportional.

The hypothesis for the inequality on the arithmetic and geometric means is that the numbers are all positive. The numbers in the Chauchy-Schwarz Inequality need not be positive. In fact,

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

is unaltered by changes in the signs of the a_i and b_i, while

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2$$

is largest when all the signs are positive.

Problems for Sections 10.3 and 10.4

1. Given that $a, b, c, d, x, y, z,$ and w are positive real numbers, prove the following [from (a) to (z)]:

   (a) If $x + y = 2$, then $xy \leq 1$.
   (b) If $xyz = 1$, then $x + y + z \geq 3$.
   (c) If $xyz = 1$, then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3$.

(d) If $x + y + z = 1$, then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9$.

(e) $\left(\frac{a+b+c}{3}\right)^3 \geq abc$.

(f) $(a+b+c+d)^4 \geq 256abcd$.

(g) $(x+y)(x-y)^2 \geq 0$.

(h) $x^3 + y^3 \geq x^2y + xy^2$.

(i) $x^4 + y^4 \geq x^3y + xy^3 \geq 2x^2y^2$.

(j) $x^5 + y^5 \geq x^4y + xy^4 \geq x^3y^2 + x^2y^3$.

(k) $x + \frac{1}{x} \geq 2$.

(l) $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$.

(m) $\frac{x}{y} + \frac{x}{z} + \frac{y}{z} + \frac{y}{x} + \frac{z}{x} + \frac{z}{y} \geq 6$.

(n) $xy(x+y) + yz(y+z) + zx(z+x) \geq 6xyz$.

(o) $\frac{x}{y} + \frac{y}{z} + \frac{z}{w} + \frac{w}{x} \geq 4$.

(p) $a + b + c \geq \sqrt{bc} + \sqrt{ca} + \sqrt{ab}$.

(q) $3(a+b+c+d) \geq 2\left(\sqrt{ab} + \sqrt{ac} + \sqrt{ad} + \sqrt{bc} + \sqrt{bd} + \sqrt{cd}\right)$.

(r) $(x+y)(y+z)(z+x) \geq 8xyz$.

(s) $[(x+y)(x+z)(x+w)(y+z)(y+w)(z+w)]^2 \geq 4096(xyzw)^3$.

(t) If $x + y + z = 1$, then $(1-z)(1-x)(1-y) \geq 8xyz$.

(u) If $x + y + z = 1$, then $\left(\frac{1}{x} - 1\right)\left(\frac{1}{y} - 1\right)\left(\frac{1}{z} - 1\right) \geq 8$.

(v) $(y+z+w)(x+z+w)(x+y+w)(x+y+z) \geq 81xyzw$.

(w) If $x + y + z + w = 1$, then $(1-x)(1-y)(1-z)(1-w) \geq 81xyzw$.

(x) $(ab+xy)(ax+by) \geq 4abxy$.

(y) $[(ab+cd)(ac+bd)(ad+bc)]^2 \geq 64(abcd)^3$.

(z) If $x + y = 1$, then $x^2 + y^2 \geq \frac{1}{2}$.

2. Given that a, b, and c are positive real numbers, show that

$$(a^2b + b^2c + c^2a)(a^2c + b^2a + c^2b) \geq 9a^2b^2c^2.$$

Is this true for all real numbers a, b, and c?

3. Show that if $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n.$$

4. Let $a_1, a_2, \ldots, a_{n-1}, a_n$ be positive, and let $a_i, a_j, \ldots, a_h, a_k$ be a permutation of these n numbers. Show that $\frac{a_1}{a_i} + \frac{a_2}{a_j} + \cdots + \frac{a_{n-1}}{a_h} + \frac{a_n}{a_k} \geq n.$

5. Let a, b, x and y be real numbers, with $a^2 + b^2 = 1$ and $x^2 + y^2 = 1$. Show that:

   (a) $(ax + by)^2 \leq 1.$
   (b) $(ax - by)^2 \leq 1.$
   (c) $-1 \leq ax + by \leq 1.$
   (d) $-1 \leq ax - by \leq 1.$

6. Let a, b, c, x, y, and z be real numbers with $a^2 + b^2 + c^2 = 1 = x^2 + y^2 + z^2$. Show that:

   (a) $(ax + by + cz)^2 \leq 1.$
   (b) $-1 \leq ax + by + cz \leq 1.$

7. Let a, b, c, d, and e be real numbers. Show the following:

   (a) $a^2 + b^2 \geq 2ab.$
   (b) $a^2 + b^2 + c^2 \geq bc + ac + ab.$
   (c) $3(a^2 + b^2 + c^2 + d^2) \geq 2(ab + ac + ad + bc + bd + cd).$
   (d) $2(a^2 + b^2 + c^2 + d^2 + e^2) \geq a(b + c + d + e) + b(c + d + e) + c(d + e) + de.$

8. Show that $a^2 + b^2 + 1 \geq b + a + ab$ for all real numbers a and b.

9. Show that if x and y are positive real numbers with $x + y = 1$, then

$$\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 \geq \frac{25}{2}.$$

10. Show that if x, y and z are positive real numbers with $x + y + z = 1$, then

$$\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 + \left(z + \frac{1}{z}\right)^2 \geq \frac{100}{3}.$$

11. Let a, b, and c be positive real numbers. Show that $\sqrt{3(bc + ca + ab)} \leq a + b + c.$

12. Let a, b, c, and d be positive. Show that

$$2\sqrt{6(ab + ac + ad + bc + bd + cd)} \leq 3(a + b + c + d).$$

13. Let A_n, G_n, and H_n be the arithmetic, geometric, and harmonic mean, respectively, of positive numbers $a_1, a_2, \ldots, a_n$. Assuming $A_n \geq G_n$, show that $A_n \geq G_n \geq H_n$.

14. Show that $n^n \geq 1 \cdot 3 \cdot 5 \cdots (2n - 1)$.

15. Show that $(1^k + 2^k + \cdots + n^k)^n \geq n^n (n!)^k$ for all positive integers n and k.

16. Show the following:

(a) $(n + 1)^n \geq 2 \cdot 4 \cdot 6 \cdots (2n)$.

(b) $n^n \left(\dfrac{n + 1}{2} \right)^{2n} \geq (n!)^3$ for all positive integers n.

17. Show the following:

(a) $n \cdot 1 + (n - 1) \cdot 2 + (n - 2) \cdot 3 + \cdots + 2(n - 1) + 1 \cdot n = \dbinom{n + 2}{3}$.

(b) $\left[\dfrac{(n + 1)(n + 2)}{6} \right]^{n/2} \geq n!$.

18. Show the following:

(a) $1 + 2 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$.

(b) $2^n \geq 1 + n\left(\sqrt{2}\right)^{n-1}$ for all positive integers n.

19. Do the following:

(a) Show that $\left(1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} \right)^n \geq \dfrac{n^n}{\sqrt{n!}}$.

(b) Show that $\sqrt{n + 1} - \sqrt{n} > \dfrac{1}{2\sqrt{n + 1}}$ for all positive integers n.

(c) Show by mathematical induction that $2\sqrt{n} > 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}}$.

(d) Show that $n! > \left(\frac{n}{4}\right)^n$.

20 Show that $\prod_{k=0}^{n} \binom{n}{k} \leq \left(\frac{2^n - 2}{n - 1}\right)^{n-1}$ for $n \geq 2$.

21. Do the following:

(a) Find the arithmetic mean of $a_1, a_2, ..., a_{100}$, given that $a_1 = 1$ and $a_2 = a_3 = ... = a_{100} = 100/99$.

(b) Prove that $\left(\frac{100}{99}\right)^{99/100} < \frac{101}{100}$.

(c) Prove that $\left(1 + \frac{1}{99}\right)^{99} < \left(1 + \frac{1}{100}\right)^{100}$.

(d) Prove that $\left(1 + \frac{1}{n}\right)^{n} < \left(1 + \frac{1}{n+1}\right)^{n+1}$ for all positive integers n.

22. Do the following:

(a) Find the arithmetic mean of $a_0, a_1, a_2, ..., a_{100}$, given that $a_0 = 1$ and $a_1 = a_2 = ... = a_{100} = 99/100$.

(b) Prove that $100^{201} > 99^{100} \cdot 101^{101}$.

(c) Prove that $\left(1 + \frac{1}{n}\right)^{n+1} > \left(1 + \frac{1}{n+1}\right)^{n+2}$ for all positive integers n.

23. Do the following:

(a) Find the arithmetic and geometric means of the roots of $x^4 - 8x^3 + 18x^2 - 11x + 2 = 0$, given that all the roots are positive.

(b) Given that all the roots of $x^6 - 6x^5 + ax^4 + bx^3 + cx^2 + dx + 1 = 0$ are positive, find $a, b, c,$ and d.

(c) Find all the roots of $x^{11} - 11x^{10} + ... - 1 = 0$, given that each root is positive.

24. Given tht a, b, and c are the lengths of the sides of a triangle, show that

$$3(bc + ac + ab) \leq (a + b + c)^2 < 4(bc + ac + ab).$$

25. For all real numbers a, b, c, x, y, and z show that

$$\sqrt{a^2 + b^2 + c^2} + \sqrt{x^2 + y^2 + z^2} \geq \sqrt{(a + x)^2 + (b + y)^2 + (c + z)^2}.$$

26. For all real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$, show that

$$\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} + \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2} \geq \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2 + \cdots + (a_n + b_n)^2}.$$

27. Show that if a and b are positive real numbers and m and n are positive integers, then

$$\frac{m^m n^n}{(m + n)^{m+n}} \geq \frac{a^m b^n}{(a + b)^{m+n}}.$$

28. Let F_n and L_n be the nth Fibonacci and nth Lucas number, respectively. Prove that

$$\left(\frac{F_{4n}}{n}\right)^n > L_2 L_6 L_{10} \cdots L_{4n-2}$$

for all integers $n \geq 2$.